Evolution of anisotropic structures and turbulence in the multidimensional Burgers equation

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(Received 20 August 2008; revised manuscript received 10 February 2010; published 22 April 2010)

The goal of the present paper is the investigation of the final evolution of anisotropic regular structures and turbulence at large Reynolds number in the multidimensional Burgers equation. We show that we have local isotropization at small scales of the velocity and potential fields inside cellular zones. For periodic waves, we have simple decay inside a frozen structure. The global structure at large times is determined by the initial correlations and for short range correlated fields we have isotropization of turbulence. Finally, we consider the final behavior of the field, when the processes of nonlinear beating interactions become frozen, and the evolution of the field is determined only by the linear dissipation.

DOI: 10.1103/PhysRevE.81.046312 PACS number(s): 47.27.Gs, 05.45.−a, 43.25.+y

I. INTRODUCTION

The well-known Burgers equation in one dimension describes a variety of nonlinear wave phenomena arising in the theory of wave propagation, acoustics, plasma physics. and so on (see, e.g., [1–6]). This equation was originally introduced by Burgers as a model of hydrodynamical turbulence [7,8]. It shares a number of properties with the Navier-Stokes equations: the same type of nonlinearity, of invariance groups and of energy-dissipation relation, the existence of a multidimensional version, etc [9]. However, Burgers equation is known to be integrable and therefore lacks the property of sensitive dependence on the initial conditions. Nevertheless, the differences between the Burgers and Navier-Stokes equations are as interesting as the similarities [10] and this is also true for the multidimensional Burgers equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \nu\nabla^2\mathbf{v}. \tag{1}$$

With external random forces the multidimensional Burgers equation is widely used as a model of randomly driven Navier-Stokes equation without pressure [11–15]. The three-dimensional form of Eq. (1) has been used to model the formation of the large scale structure of the Universe when pressure is negligible. Known as "adhesion" approximation this equation describes the nonlinear stage of gravitational instability arising from random initial density fluctuations [16–20]. Other problems leading to the multidimensional Burgers equation, or variants of it, include surface growth under sputter deposition and flame front motion [21,22]. For the deposition problem, the velocity in the multidimensional Burgers equation $\mathbf{v}=-\nabla\psi$ is the gradient of the surface and the mean-square gradient $E(t)=\langle(\nabla\psi(\mathbf{x},t))^2\rangle=\langle\mathbf{v}^2(\mathbf{x},t)\rangle$, which is a measure of the roughness of the surface, may either decrease or increase with time. In such instances, the velocity potential $\psi(\mathbf{x},t)$ corresponds to the shape of the front's surface, and the equation for ψ is identical to the KPZ (Kardar, Parisi, Zhang) equation [4,21,23,24].

When the initial potential $\psi_0(\mathbf{x})\equiv\psi(\mathbf{x},0)$ is a superposition of one dimensional potentials $\psi_{0,i}(x_i)$, namely, $\psi_0(\mathbf{x})=\Sigma_i\psi_{0,i}(x_i)$ and $v_{0,i}(\mathbf{x})=v_{0,i}(x_i)$ where the sum over i runs over different directions of space, there is no interaction between the velocity components $v_{0,i}(\mathbf{x},t)=v_{0,i}(x_i,t)$, so that the evolution of each component becomes determined by a one-dimensional Burgers equation. But before dealing with the evolution of fields in the multidimensional Burgers equation, we now discuss very shortly the evolution of simple types of velocity fields in the one-dimensional Burgers equation [1–4], and use these to study the behavior of "plane" orthogonal waves in two-dimensional Burgers equation with different initial spatial scales.

In the limit of very small viscosities $\nu\to 0$ and thus for Reynolds numbers $\mathrm{Re}\equiv vL/\nu$ going to infinity, a monochromatic initial condition $v_0(x)=k_0\psi_0\sin k_0x$ [$\psi_0(x)=\psi_0\cos k_0x$], is transformed at $t\gg t_{\mathrm{nl}}=1/k_0^2\psi_0$ into saw-tooth wave with gradient $\partial v/\partial x=1/t$ and the same period $L_0=2\pi/k_0$. It is important that at this stage the amplitude $a=L_0/t$ and the energy $E(t)=L_0^2/12t^2$ do not depend any more on the initial amplitude. Thus, if we compare the evolution of two components $v_{0,i}(x_i,t)$ with equal initial potential ψ_0 and different scales L_i ($L_1\ll L_2$), the initial energy will be much higher for the component with smaller scale $E_1(0)/E_2(0)=L_2^2/L_1^2\gg 1$. But asymptotically, we will have the inverse situation $E_1(t)/E_2(t)\to L_1^2/L_2^2\ll 1$. For large but finite initial Reynolds number $\mathrm{Re}_0=\psi_0/2\nu$ the shock fronts have a finite width $\sim\nu t/L_0$ and at $t\gg t_{\mathrm{nl}}\,\mathrm{Re}_0$ we will reach a linear stage of evolution where $v(x,t)=4\nu k_0\sin(k_0x)\exp(-\nu k_0^2t)$.

Continuous random initial fields are also transformed into sequences of regions (cells) with the same gradient $\partial v/\partial x=1/t$, but with random position of the shocks separating them. The merging of the shocks leads to an increase in the integral scale of turbulence $L(t)$ and because of this the energy $E(t)\sim L^2(t)/t^2$ of random field decreases more slowly than the energy of periodic signal. The type of the turbulence

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evolution is determined by the behavior of the large scale part of the initial energy spectrum $E_0(k) \sim \alpha^2 k^n$. For $n<1$, the initial potential is Brownian or fractional Brownian motion and scaling argument may be used [3,4,8,19,25,26]. In this case, the turbulence in self-similar and with integral scale $L(t)=(\alpha t)^{2/3+n}$. For $n>1$ the law of energy decay strongly depends on the statistical properties of the initial field [4,27–31]. For an initially Gaussian velocity field, the integral scale $L(t)$ and the energy of the turbulence $E(t)$

$$L(t) = (t\sigma_\psi)^{1/2} \ln^{-1/4}\left(\frac{t\sigma_\psi}{2\pi l_0^2}\right),$$

$$E(t) = L^2(t)/t^2 \tag{2}$$

are determined only by two integral characteristics of the initial spectrum : the variance of the initial potential $\sigma_\psi^2 = \langle \psi_0^2 \rangle$ and the velocity variance $\sigma_v^2 = \langle v_0^2 \rangle$ [3,8,32–37]. Here, $l_0 = \sigma_\psi / \sigma_v$ is the integral scale of the initial perturbation, and $\sigma_\psi / l_0^2 = t_{\mathrm{nl}}$ is the nonlinear time. Thus, the energies of two components with equal initial potential variance σ_ψ but with different scales $l_{0,i}, (l_{0,1} \ll l_{0,2})$, that are very different at $t=0$ $E_1(0)/E_2(0) = l_2^2/l_1^2 \gg 1$, can become equal $E_1(t)/E_2(t) \simeq 1$ for very large times due to the logarithmic correction. For large but finite Reynolds number $\mathrm{Re}_0 = \sigma_\psi / 2\nu$, the shock fronts have a finite width $\sim \nu t / L(t)$ and due to the multiple merging of shocks the linear regime take place at very large times $t \gg t_{\mathrm{nl}} \exp(\mathrm{Re}_0^2)/\mathrm{Re}_0$. In the linear stage, the energy decays as $Ct^{-3/2}$, where $C \sim L_0 \exp(\mathrm{Re}_0^2)/\mathrm{Re}_0$.

Now, the goal of the present paper is the investigation of the evolution of anisotropic structures and turbulence at large Reynolds number in the multidimensional case, when we have nonlinear interactions between the harmonic components in the different directions. We show that we have local isotropization of the velocity and of the potential fields inside a cellular structure. In the case of periodic waves, we have the decay of this frozen structure. For random initial conditions, the global final structure will be determined by the long range correlations of the initial field; in the short-range correlation case we have final global isotropization of turbulence. The other limit we will consider in this paper is the large time behavior of the field, when the processes of nonlinear self-action and harmonic interaction become frozen, and the evolution of the field is determined solely by the linear dissipation.

The paper is organized as follows. In Sec. II, we formulate our problem and list some results about the solution of multidimensional Burgers equation in the limit of vanishing viscosity and its long time behavior. We also show that we have local isotropization of the velocity and potential fields. In Sec. III, we consider the interaction of plane waves and evolution of periodic structures in two-dimensional Burgers equation. In Sec. IV, we consider the evolution of anisotropic multidimensional Burgers turbulence in the inviscid limit. We also discuss here the influence of finite viscosity and long range correlations on the late stage evolution of Burgers' turbulence.

II. MULTIDIMENSIONAL BURGERS EQUATION, THE LIMIT OF VANISHING VISCOSITY AND THE LARGE TIME BEHAVIOR

We shall study the initial value problem for the unforced multidimensional Burgers Eq. (1) and consider only the potential solution of this equation, namely,

$$\mathbf{v}(\mathbf{x},\mathbf{t}) = -\nabla\psi(\mathbf{x},\mathbf{t}). \tag{3}$$

The velocity potential $\psi(\mathbf{x},t)$ satisfies the following nonlinear equation:

$$\frac{\partial\psi}{\partial t} = \frac{1}{2}(\nabla\psi)^2 + \nu\nabla^2\psi. \tag{4}$$

The equation for the velocity potential ψ is identical to the equation [4,21,23], which is usually written in the terms of the variable $h = \lambda^{-1}\cdot\psi$. The parameter λ has the dimension of length divided by time and is the local velocity of the surface growth. Henceforth, $h(\mathbf{x},\mathbf{t})$ has the dimension of length and is the measure of the surface's smoothness. In this case $\mathbf{v}=-\nabla\psi$ is the gradient of the surface. The roughness of the surface is measured by its mean-square gradient

$$E(t) = \langle (\nabla\psi(\mathbf{x},t))^2 \rangle = \langle \mathbf{v}^2(\mathbf{x},t) \rangle = \sum_i E_i(t), \tag{5}$$

Here, angular brackets denote ensemble averages or space averages for periodic structures. For one-dimensional homogeneous fields, $E(t)$ is the energy density and always decreases with time. At the initial stage of evolution and in the limit of vanishing viscosity, the multidimensional Burgers equation corresponds to the free motion of initially spatially uniformly distributed independent particles having initial velocities $\mathbf{v}(\mathbf{x},t)$. In Lagrangian representation, the velocity of the particle $\mathbf{V}(t;\mathbf{y})$ is thus a constant. Here, $\mathbf{y}$ is the initial (Lagrangian) coordinate of the particle. In the one-dimensional case, the increasing in the length of an elementary Eulerian interval $\Delta x = \Delta y + t\Delta V$ is compensated by the decreasing in length $\Delta x = \Delta y - t\Delta V$ of the steepening interval and therefore the energy of the wave (the mean roughness of the curve) is conserved. After shock formation(s), the energy of the wave decreases with time. In the multidimensional case, the modification of the elementary Eulerian volume depends also on the initial curvature of the perturbation and we do not generally have compensation of steepening and stretching volumes. Thus, for $d>1$ the roughness of the surface, measured by its mean-square gradient $E(t)$ [see Eq. (5)] may either decrease or increase with time [27,31]. The increase of the mean-square gradient in the multidimensional Burgers equation (in contrast with $d=1$) comes from the nonconservative form of that equation. Nevertheless, we will use the expression "turbulence energy" for the value of $E(t)$ and call $E_i(t) = \langle (\partial\psi/\partial x_i)^2 \rangle = \langle v_i^2 \rangle$ the energy of the i-th velocity component.

Using the Hopf-Cole transformation [38,39],

$$\psi(\mathbf{x},t) = 2\nu \log U(\mathbf{x},t), \tag{6}$$

one can reduce Eq. (1) to the linear diffusion equation

$$\frac{\partial U}{\partial t} = \nu \nabla^2 U, \quad U(\mathbf{x},0) = U_0(\mathbf{x}) = \exp\left[\frac{\psi_0(\mathbf{x})}{2\nu}\right]. \tag{7}$$

The goal of the present paper will be the investigation of the evolution of regular structures and turbulence at large Reynolds number. So, at least for not very large times, we can use the solutions of Burgers equation in the limit of vanishing viscosity. The other limit we will consider is the long time behavior of the field, when the processes of nonlinear self-action and harmonic interaction become frozen, and the evolution of the field is determined only by the linear dissipation. In that case we will use the linearization via the Hopf-Cole transformation (6).

A. Limit of vanishing viscosity

In the limit of vanishing viscosity $\nu \to 0$, use of the Laplace's method leads to the following "maximum representation" for the potential velocity field [3,19,38]

$$\psi(\mathbf{x},t) = \max_{\mathbf{y}} \Phi(\mathbf{x},\mathbf{y},t), \tag{8}$$

$$\Phi(\mathbf{x},\mathbf{y},t) = \psi_0(\mathbf{y}) - \frac{(\mathbf{x}-\mathbf{y})^2}{2t}, \tag{9}$$

$$\mathbf{v}(\mathbf{x},t) = \frac{\mathbf{x}-\mathbf{y}(\mathbf{x},t)}{t} = \mathbf{v}_0[\mathbf{y}(\mathbf{x},t)]. \tag{10}$$

Here, $\psi_0(\mathbf{y})$ is the initial potential and $\mathbf{v}_0(\mathbf{x}) = -\nabla\psi_0(\mathbf{x})$. In Eq. (10), $\mathbf{y}(\mathbf{x},t)$ is the Lagrangian coordinate where the function $\Phi(\mathbf{x},\mathbf{y},t)$ achieves its global or absolute maximum for a given coordinate $\mathbf{x}$ and time t. It is easy to see that $\mathbf{y}$ is the Lagrangian coordinate from which starts the fluid particle which will be at the point $\mathbf{x}$ at time t [3].

At large times, the paraboloidal peaks in Eq. (9) define a much smoother function than the initial potential $\psi_0(\mathbf{y})$. Consequently, the absolute maximum of $\Phi(\mathbf{x},\mathbf{y},t)$ coincides with one of the local maxima of $\psi_0(\mathbf{y})$. In the neighborhood of any local maximum $\mathbf{y_k}$ we can represent the initial potential in the following form:

$$\psi_0(\mathbf{x}) = \psi_{0,k}\left(1 - \sum_i \frac{(x_i - y_{i,k})^2}{2L_i^2}\right), \tag{11}$$

where x_i now are the principal axes of the local quadratic form describing the potential near the local maximum, and the L_i the radii of curvature along that direction. At relatively large times, the Eulerian velocity field $\mathbf{v}(\mathbf{x},t)$ in the whole space will be determined by the particles moving away from the small regions near the nearest local maximum of $\psi_0(\mathbf{x})$

$$y_i(\mathbf{x},t) = \frac{(x_i - y_{i,k})}{(1 + \psi_{0,k} t/L_i^2)}. \tag{12}$$

Then, the Lagrangian coordinate $\mathbf{y}(\mathbf{x},t)$ becomes a discontinuous function of $\mathbf{x}$, constant within cells centered around each local maximum of $\psi_0(\mathbf{x})$, but jumping at the boundaries between cells [3,19]. The velocity field $\mathbf{v}(\mathbf{x},t)$ has discontinuities (shocks) and the potential field $\psi(\mathbf{x},t)$ has gradient discontinuities (cusps) at the cell boundaries. From Eqs. (10) and (12), it becomes clear that inside the cells the velocity and potential fields have a universal isotropic and self-similar structure

$$\psi(\mathbf{x},t) = \psi_0(\mathbf{y}_k) - \frac{(\mathbf{x}-\mathbf{y}_k)^2}{2t}. \tag{13}$$

$$\mathbf{v}(\mathbf{x},t) = \frac{\mathbf{x}-\mathbf{y}_k}{t}. \tag{14}$$

One can see that due to the nonlinearity, there is local isotropization of the velocity field in the neighborhood of every local maximum of $\psi_0(\mathbf{x})$. The longitudinal (perpendicular to the cell boundaries) component of the velocity vector $\mathbf{v}(\mathbf{x},t)$ consists of a sequence of saw-tooth pulses, just as in one dimension. The transverse component(s) consist of sequences of rectangular pulses. At large times, the global structure and evolution of the velocity and potential fields will be determined by the properties of local maxima $\psi_0(\mathbf{y}_k)$. In the case of random initial conditions, cell wall motion results in continuous change in cell shape with cells swallowing their neighbors and thereby inducing growth of the external scale $L(t)$ of the Burgers turbulence.

B. Long time limit of the solution

Let us now discuss the long time limit of the solution of Burgers equation. Consider a group of perturbation with a bounded initial potential $\langle \psi_0(\mathbf{x})^2 \rangle < \infty$ assuming that $\psi_0(\mathbf{x})$ is a periodic structure or homogeneous noise with rapidly decreasing probability distribution of the potential values ψ_0. For such a perturbation in $U(\mathbf{x},t)$ we separate a constant component $\bar{U}$

$$U(\mathbf{x},t) = \bar{U} + \tilde{U}(\mathbf{x},t) = \bar{U}[1 + u(\mathbf{x},t)]. \tag{15}$$

Here, $u(\mathbf{x},t) = \tilde{U}(\mathbf{x},t)/\bar{U}$ is the relative perturbation of field $U(\mathbf{x},t)$. Inserting Eq. (15) into Eq. (7) we see that $\bar{U}$ does not depend on time. Here, $\tilde{U}_0(\mathbf{x})$ and $u_0(\mathbf{x})$ are fields with zero mean value (over the period or statistically in the noise case). As time goes on, the viscous dissipation and smoothing of the inhomogeneities causes the amplitude (variance) of the field $\tilde{U}(\mathbf{x},t)$ to decrease. At times when its relative amounts $\tilde{U}$ become small in comparison with $\bar{U}$ ($|u| \ll 1$) the solution (6) will become equal to

$$\mathbf{v}(\mathbf{x},t) = -2\nu \nabla \tilde{U}(\mathbf{x},t)/\bar{U} = -2\nu \nabla u(\mathbf{x},t). \tag{16}$$

As $\tilde{U}(\mathbf{x},t)$ and $u(\mathbf{x},t)$ satisfy the linear diffusion equation, then $\mathbf{v}(\mathbf{x},t)$ also at these times obeys the linear equation. This is precisely the definition of the beginning of the linear stage. The complete accumulated nonlinear effects are described in this solution by the nonlinear integral relation between the initial velocity field $\mathbf{v}_0(\mathbf{x})$ and the fields $\tilde{U}(\mathbf{x},0)$, $\bar{U}$ Eqs. (3) and (7), and are characterized by the value $|\Delta\psi_0|/\nu \sim \mathrm{Re}_0$. Here, $\Delta\psi_0$ is the characteristic change in amplitude of ψ_0, and Re_0 is the initial Reynolds number.

From Eq. (16), it is easy to obtain the well known result that for $\mathrm{Re}_0 \gg 1$, initial harmonic conditions asymptotically

keep their harmonic form, but with amplitudes not depending any more on the initial amplitudes [1,2]. At large initial Reynolds number, homogeneous Gaussian field $v_0(x)$ transforms in the nonlinear stage into a series of saw-tooth waves with strong non-Gaussian statistical properties [3,35]. Nevertheless at very large time, when the relation (16) is valid, the distribution of the random field $\mathbf{v}(\mathbf{x},t)$ with statistically homogeneous initial potential $\psi_0(\mathbf{x})$ converges weakly to the distribution of homogeneous Gaussian random field with zero mean value [27]. This stage of evolution is known as the Gaussian scenario in Burgers turbulence. In the absence of long correlations in the initial potential field the potential (and velocity consequently) will thus have a universal covariance function [3,27]. But the amplitude of this function is nonlinearly related to the initial covariance function of the field $\psi_0(\mathbf{x})$ and increases proportionally to $\exp(\mathrm{Re}_0^2)$ with increasing in initial Reynolds number Re_0. When the initial potential has long-range correlations $(\langle\langle(\psi_0(\mathbf{x})\psi_0(0)\rangle = |\mathbf{x}|^{-\alpha}F(\mathbf{x}/x)$, $0<\alpha<3$, at large x), we will have conservation of these in the linear stage as long correlation in the anisotropy of the field $F(\mathbf{x}/x)$ [27].

III. EVOLUTION OF PERIODIC STRUCTURES AND THE INTERACTION OF PLANE WAVES IN TWO-DIMENSIONAL BURGERS EQUATION

It was shown in the previous section that at large time we will reach a cellular structure of the field with universal behavior of potential and velocity inside each cell. The global structure of the field will be determined by the properties of the local maxima of initial potential.

Let us consider the evolution of a periodic structure

$$\psi_0(x_1,x_2) = 2\psi_0 \cos(k_2x_2)\cos(k_1x_1) \tag{17}$$

in the two-dimensional Burgers equation. The evolution of this structure can also be interpreted as the interaction of two plane waves with equal wave number modulus k_0

$$\psi_0(x_1,x_2) = \psi_0 \cos(k_1x_1 + k_2x_2) + \psi_0 \cos(k_1x_1 - k_2x_2). \tag{18}$$

Here, $k_1=k_0n_1=2\pi/l_1$, $k_2=k_0n_2=2\pi/l_2$ and $\mathbf{n}$ is a unit vector with the component n_1,n_2. For each of these noninteracting plane waves, the energy of the i-th component is $E_{i,p}(t)=E(t)n_i^2$, where $E(t)$ is the energy of the plane wave and $E(0)=\psi_0^2k_0^2/2$.

For $\nu\to 0$ the plane wave will be transformed at $t\gg t_{\mathrm{nl}}=1/k_0^2\psi_0$ into a saw-tooth wave with gradient $\partial v/\partial x=1/t$ and energy $E(t)=\pi^2/3k_0^2t^2$. For large but finite Reynolds number, we will reach a final linear stage of evolution where $v(x,t)=4\nu k_0 \sin(k_0x)\exp(-\nu k_0^2t)$. We now consider these two cases one after the other.

A. Final stage of evolution at infinite Reynolds number

Consider first the late stage of evolution of the periodic structure at times $t\gg t_{\mathrm{nl}}$. At this stage the velocity has the universal form (14) in each cell, where $\mathbf{y_k}$ are the maxima of the initial potential (17). For this particular initial potential, there are two sets of maxima of equal value corresponding to

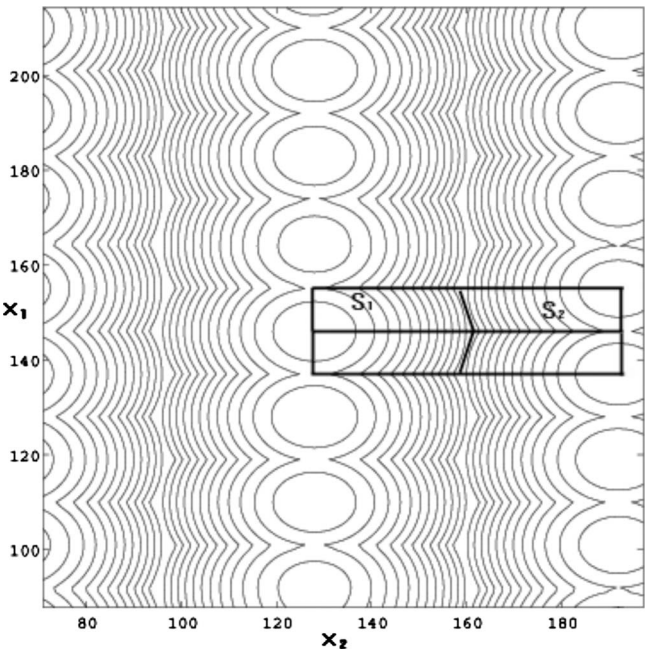

FIG. 1. An example of a two-dimensional cellular structure observed when looking at the isovalues of the potential $\psi(x_1,x_2)$ when $k_1\gg k_2$.

the conditions $\cos k_1x_1=\cos k_2x_2=1$ and $\cos k_1x_1=\cos k_2x_2=-1$. The shock lines (cell boundaries) of the velocity field do not move in time, and they are situated on the midpoint and orthogonal to the vectors connecting each local maximum to its nearest neighbors, thus defining a cellular structure with the local maxima at the cell centers. Due to the symmetry of initial conditions, the velocity field is symmetric over the point $(l_1/2,l_2/2)$. Assume now that $l_1\le l_2$ and consider the velocity fields inside the region S: $(x_1\in[0,l_1/2],x_2\in[0,l_2/2])$, see Fig. 1.

The region S is divided by the shock line in the regions S_1 and S_2

$$S_1:\left(0\le x_2\le -\frac{l_1}{l_2}x_1+\frac{l_1^2+l_2^2}{4l_2^2}\right). \tag{19}$$

The center of the cell inside the region S_1 is at the point $(x_1=0,\ x_2=0)$ and inside region S_2, it is at the point $(x_1=l_1/2,\ x_2=l_2/2)$. Consequently for the velocity fields, one can get

$$v_1=x_1/t,\quad v_2=x_2/t,\quad \mathbf{x}\in S_1,$$

$$v_1=(x_1-l_1/2)/t,\quad v_2=(x_2-l_2/2)/t,\quad \mathbf{x}\in S_2. \tag{20}$$

Thus, at large time, we have a frozen structure of the field with decreasing amplitude $\sim t^{-1}$. For the energy of the velocity components, we obtain from expressions (20)

$$E_1(t)=\frac{l_1^2}{12t^2}\left(1-\frac{l_1^2}{l_2^2}\right)=\frac{\pi^2}{3k_0^2n_1^2t^2}\left(1-\frac{n_2^2}{n_1^2}\right),$$

$$E_2(t) = \frac{l_2^2}{48t^2}\left(1 + \frac{l_1^4}{l_2^4}\right) = \frac{\pi^2}{12k_0^2 n_2^2 t^2}\left(1 + \frac{n_2^4}{n_2^1}\right). \quad (21)$$

From Eq. (20), we get that in the case of very anisotropic fields ($l_1 \ll l_2$), ($n_1 \simeq 1, n_2 \ll 1$) the velocity component v_1 will reproduce the behavior of the velocity in the one-dimensional Burgers equation, but the large scale component will now have period $L=l_2/2$ instead of the initial $L=l_2$. Comparing now the decay of the periodic structure (17), which is a superposition of two plane waves, with the energy decay $E_{i,p}$ that each component would have in the one-dimensional case, we see that for $l_1 \ll l_2$, the energy of the small scale $E_1 \simeq \pi^2/3k_0^2 n_1^2 t^2 \simeq E_{i,1} = \pi^2 n_1^2/3k_0^2 t^2$ decays similarly, but the energy of the large scale component $E_2 \simeq \pi^2/12k_0^2 n_2^2 t^2 \gg E_{i,2} = \pi^2 n_2^2/3k_0^2 t^2$ has a much slower evolution. This shows that the two components do *not* evolve independently in the two-dimensional case.

B. final stage of evolution at finite Reynolds number

Consider now the linear stage of the evolution of the periodic structure, when the Hopf-Cole solution is reduced to the linear relation (16) between the velocity field $\mathbf{v}(\mathbf{x},\mathbf{t})$ and the solutions $\tilde{U}(\mathbf{x},t)$, $u(\mathbf{x},t)$ of the linear diffusion Eq. (7). Using the relation $\exp(z\cos\theta) = I_0(z) + 2\Sigma_{m=1}^{\infty} I_m(z)\cos m\theta$, where $I_m(z)$ are modified Bessel functions, we can write the solution of this equation in the form (7) and (17)

$$\begin{aligned} U(\mathbf{x},t) = I_0^2(\mathrm{Re}_0) 2\sum_{m=1}^{\infty} & I_0(\mathrm{Re}_0) I_m(\mathrm{Re}_0) \\ & \times \{\cos[m(k_1 x_1 + k_2 x_2)] + \cos[m(k_1 x_1 - k_2 x_2)]\} \\ & \times e^{-\nu m^2 (k_1^2 + k_2^2)t} + 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} I_m(\mathrm{Re}_0) I_n(\mathrm{Re}_0) \\ & \times \{\cos[(n+m)k_1 x_1 + (n-m)k_2 x_2] \\ & \times e^{-\nu[(n+m)^2 k_1^2 + (n-m)^2 k_2^2)]t} + \cos[(n-m)k_1 x_1 \\ & + (n+m)k_2 x_2)] e^{-\nu[(n-m)^2 k_1^2 + (n+m)^2 k_2^2)]t}\}, \end{aligned} \quad (22)$$

where $\mathrm{Re}_0 = \psi_0/2\nu$. Here, the first sum describes the nonlinear evolution of the two plane waves and the double summation is the interaction between the plane waves. From Eq. (22), we find that the constant component appearing in Eq. (15) is $\bar{U} = I_0^2(\mathrm{Re}_0)$ and $\tilde{U}(\mathbf{x},t) = U(\mathbf{x},t) - \bar{U}$. In the linear stage of evolution, when $u(\mathbf{x},t) = \tilde{U}(\mathbf{x},t)/\bar{U} \ll 1$ we have from Eqs. (6) and (15)

$$\psi(\mathbf{x},t) = \bar{\psi} + \tilde{\psi}(\mathbf{x},t), \quad \tilde{\psi}(\mathbf{x},t) \simeq 2\nu u(\mathbf{x},t), \quad (23)$$

where $\bar{\psi} = 2\nu \log \bar{U}$. The asymptotic behavior of the potential (shape of the surface) $\tilde{\psi}(\mathbf{x})$ will be determined by the lower frequencies of the decaying exponential solution (22). For $3^{1/2}k_2 > k_1 > k_2$, we have

$$\tilde{\psi}(\mathbf{x},t) \simeq 4\nu[I_1(\mathrm{Re}_0)/I_0(\mathrm{Re}_0)] \times \cos(k_1 x_1)\cos(k_2 x_2) e^{-\nu(k_1^2 + k_2^2)t} \quad (24)$$

and consequently the velocity components v_i decay independently like the i-th component of a plane wave

$$\begin{aligned} v_1(\mathbf{x},t) = {} & 4\nu k_1 (I_1(\mathrm{Re}_0)/I_0(\mathrm{Re}_0)) \\ & \times \sin(k_1 x_1)\cos(k_2 x_2) e^{-\nu(k_1^2 + k_2^2)t}, \end{aligned} \quad (25)$$

$$\begin{aligned} v_2(\mathbf{x},t) = {} & 4\nu k_2 (I_1(\mathrm{Re}_0)/I_0(\mathrm{Re}_0)) \\ & \times \sin(k_2 x_2)\cos(k_1 x_1) e^{-\nu(k_1^2 + k_2^2)t}, \end{aligned} \quad (26)$$

which is equal to the linear solution of the one-dimensional Burgers equation for small Reynolds numbers.

But for $3^{1/2}k_2 < k_1$, the nonlinear interaction between the plane waves will change the asymptotic evolution of the potential (shape of the surface). Now the leading term in Eq. (22) is, along axis x_1

$$\tilde{\psi}(\mathbf{x},t) \simeq 2\nu(I_2(\mathrm{Re}_0)/I_0(\mathrm{Re}_0)) \times \cos(2k_2 x_2) e^{-\nu(4k_2^2)t} \quad (27)$$

and will be the second harmonic for axis x_2,

$$v_2(\mathbf{x},t) = 4\nu k_2 (I_2(\mathrm{Re}_0)/I_0(\mathrm{Re}_0)) \times \sin(2k_2 x_2) e^{-\nu(4k_2^2)t} \quad (28)$$

and so we have depression of the amplitude of the potential along the axis x_1. The evolution of the parallel gradient on the surface along x_1, which is the velocity component v_1, is still determined by Eq. (25) and so it will decay faster than the transverse gradient v_2 along x_2.

Thus, due to the nonlinearity which leads to the generation of cross-wave numbers, we will have at the late linear stage for the velocity component v_2 the dominance of the second harmonic of the initial spatial frequency k_2. This will be true even for small Reynolds numbers. For very large Reynolds numbers Re_0, we will have $I_1(\mathrm{Re}_0)/I_0(\mathrm{Re}_0) \simeq 1$ and the amplitude of v_1, v_2 do not depend on the amplitude of the initial perturbation.

C. Transition processes at infinite Reynolds number

Let us now consider the evolution of very anisotropic fields when the angle between interacting plane waves is small $n_1 \gg n_2$. Consider first a more general situation when a periodic plane wave initial condition is modulated by a large scale function $M(\mathbf{x})$, so that the initial potential can be written as

$$\psi_0^M(\mathbf{x}) = M(\mathbf{x})\psi_0 \cos(k_1 x_1). \quad (29)$$

We assume that the function $M(\mathbf{x})$ is characterized by the scales $L_{M,i}$ and $L_{M,i} \gg l_1 = 1/2\pi k_1$. The case that we considered before of two plane interacting waves (17) corresponds to $M(\mathbf{x}) = 2\cos(k_2 x_2)$ and we will pursue now the study of that case. The initial velocity field corresponding to Eq. (29) is thus

$$v_{1,0}(x_1, x_2) \simeq k_1 \psi_0 \sin(k_1 x_1) M(x_1, x_2), \quad (30)$$

$$v_{2,0}(x_1,x_2) = -\psi_0 \cos(k_1 x_1) M'_{x_2}(x_1,x_2). \quad (31)$$

In the limit of vanishing viscosity $\nu \to 0$ the evolution of the velocity field is described by the Eq. (10) and $\mathbf{v}(\mathbf{x},t) = \mathbf{v}_0[\mathbf{y}(\mathbf{x},t)]$, where $\mathbf{y}(\mathbf{x},t)$ is the Lagrangian coordinate from which starts the fluid particle that will be at the point $\mathbf{x}$ at time t. Because $L_{M,i} \gg l_1$, the velocity component $v_2 \ll v_1$ and for times $t \ll t_{\mathrm{nl},2} = L^2_{m,2}/\psi_0$, one can neglect the drift of the particles along x_2 axis. In that case, we get for the Lagrangian coordinates

$$X_1(y_1,y_2,t) = y_1 + tk_1\psi_0 \sin(k_1 y_1) M(y_1,x_2),$$

$$X_2(y_1,y_2,t) = y_2. \quad (32)$$

For $t < t_{\mathrm{nl},1} = 1/k_1^2\psi_0$, this solution $\mathbf{v}(\mathbf{x},t) = \mathbf{v}_0[\mathbf{y}(\mathbf{x},t)]$ is single-valued. But for $t > t_{\mathrm{nl},1}$, we need to introduce the shock to remove the multivalued solution. In a quasistatic approximation, we assume that the evolution of the velocity component $v_1(x_1,x_2,t)$ follows the one-dimensional Burgers equation with an initial harmonic perturbation $v_{1,0} = A \sin(k_1 x_1)$. The amplitude of the perturbation $A = k_1\psi_0 M(x_{1,m},x_2)$ $(x_{1,m} = l_1 m)$ now depends slowly on the coordinate x_2 as a parameter and can be taken as a constant independent of x_1 over each period of the harmonic perturbation.

Let us now consider the nonlinear stage of evolution $t_{\mathrm{nl},1} \ll t \ll t_{\mathrm{nl},2}$ when the velocity component v_1 transforms into saw-tooth waves. Consider the region when $M(x_1,x_2) > 0$. It is easy to see that for time $t \gg t_{\mathrm{nl},1}$, each period $l_1(m-1/2) < x_1 < l_1(m+1/2)$ will be reached by particles coming from a small region near the point $x_{1,m} = l_1 m$ and the solution of the equations $x_1 = X(y_1,y_2,t)$, $x_2 = X_2$ can be written as

$$y_1 - x_{1,m} = \frac{x_1 - x_{1,m}}{1 + tk_1^2\psi_0 M(x_{1,m},x_2)}, \quad y_2 = x_2. \quad (33)$$

The shocks will lie on the line $x_{1,s} = l_1(m+1/2)$. From Eqs. (32) and (33), we also get for the velocity components inside $l_1(m-1/2) < x_1 < l_1(m+1/2)$ (taking for $M > 0$)

$$v_1(x_1,x_2,t) = \frac{x_1 - y_{1,m}}{t}\left(1 - \frac{1}{1 + tk_1^2\psi_0 M(x_{1,m},x_2)}\right),$$

$$v_2(x_1,x_2,t) = -\psi_0 M'_{x_2}(x_{1,m},x_2)\cos\left(\frac{k_1 x_1}{tk_1^2\psi_0 M(x_{1,m},x_2)}\right). \quad (34)$$

From this equation, one can see that for $t \gg t_{\mathrm{nl},1}$ the velocity component $v_1(x_1,x_2,t)$ is transformed into a saw-tooth wave $v_1 \simeq (x_1 - y_{1,m})/t$ like in the one-dimensional case. This means that we have a full depression of the initial modulation amplitude of this component. On the other hand, the velocity component v_2 loses its periodic modulation along x_1 and becomes $v_{2,0}(x_1,x_2,t) \simeq -\psi_0 M'_{x_2}(x_1,x_2)$ for positive M and $v_{2,0}(x_1,x_2,t) \simeq \psi_0 M'_{x_2}(x_1,x_2)$ for negative M. Comparing with Eq. (31), we see that the energy of this component has increased by a factor of 2 with respect to the initial energy.

For this kind of two-scale initial conditions, the velocity field can also be split in the two "slow" and "fast" parts

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_l(\mathbf{x},t) + \mathbf{v}_s(\mathbf{x},t), \quad \mathbf{v}_l(\mathbf{x},t) = \langle \mathbf{v}(\mathbf{x},t) \rangle, \quad (35)$$

where the brackets $\langle \ldots \rangle$ means the averaging over period l_1. Here, $\mathbf{v}_l(\mathbf{x},t)$ is the large scale component and $\mathbf{v}_s(\mathbf{x},t)$ is the small scale component. We now assume that the evolution of the small scale component $\mathbf{v}_s$ may be described in a quasistatic approximation. The mean velocity $\mathbf{v}_l$ has a length scale of the order of $L_{M,i}$, and at the initial stage $t \ll t_{\mathrm{nl},M} = L^2_{M,i}/\sigma_\psi$ one can neglect nonlinear distortion and dissipation of this component. Then from Eq. (4), we have

$$\frac{\partial \mathbf{v}_l(\mathbf{x},t)}{\partial t} = -\frac{1}{2}\nabla \langle \mathbf{v}_s^2(\mathbf{x},t) \rangle = -\frac{1}{2}\nabla E_s(\mathbf{x},t). \quad (36)$$

The integration of this equation over t give the evident expression for the coherent component

$$\mathbf{v}_l(\mathbf{x},t) = -\nabla \langle \psi(\mathbf{x},t) \rangle \quad (37)$$

Here, we have used the Eq. (4) for the potential $\psi(\mathbf{x},t)$. From Eq. (36), we have that the generation of the large scale component is determined by the gradient of the energy of small scale component. But the periodic modulation $E_s(x,t)$ at $t \gg t_{\mathrm{nl},1}$ does not depend anymore on the initial amplitude, so that at this time there is no generation of the large scale component. The gradient of mean potential $\langle \psi(\mathbf{x},t) \rangle = \psi_0 |M(\mathbf{x})| - l_1^2/16t$ at this stage is independent of t and from Eq. (37) we get

$$\mathbf{v}_l(\mathbf{x},t) = -\nabla \langle \psi(\mathbf{x},t) \rangle = -\psi_0 \nabla |M(\mathbf{x})|. \quad (38)$$

The amplitude of the small scale component is l_1/t, while the amplitude of the large scale component is in order $\psi_0/L_{M,i}$ and it means that at $t > L_{M,i} l_1/\psi_0$ the main part of energy is in the large scale component. The nonlinear distortion of large scale component is significant at $t > \min(L^2_{M,i})/\psi_0$. The final evolution of this component now strongly depends on the properties of the modulation function $M(\mathbf{x})$.

For the periodic structure (17) when $k_1 \gg k_2$ we have that the plane wave $\psi_0 \cos(k_1 x_1)$ is modulated by the large scale function $M(\mathbf{x}) = \cos(k_2 x_2)$ (29). The initial perturbation in this case is a periodic structure with periods $l_1 = 2\pi/k_1$, $l_2 = 2\pi/k_2$ and $l_1 \ll l_2$.

Before the nonlinear distortion of large scale component $t \ll t_{\mathrm{nl},2} = l_2/\sigma_\psi$, the evolution of structure take place like in general case of modulated wave (29). The velocity component $v_1(x_1,x_2,t)$ is transformed into a saw-tooth wave (Fig. 2) and we have a fast depression of the initial modulation amplitude of this component (Fig. 2). The velocity component v_2 loses its periodic modulation over x_1 and $v_{2,0}(x_1,x_2,t) \simeq \psi_0 |\cos(k_2 x_2)|'_{x_2}$. The period of this component is one half the initial period (Fig. 3) and the energy increases by a factor 2 in comparison with the initial energy. The behavior of the energy is shown on the Fig. 4.

IV. EVOLUTION OF ANISOTROPIC MULTIDIMENSIONAL BURGERS TURBULENCE

A. Intermediate stage of evolution of an anisotropic random field

By Burgers turbulence, we mean here the Burgers problem, with random initial conditions, see [5] and, in this sec-

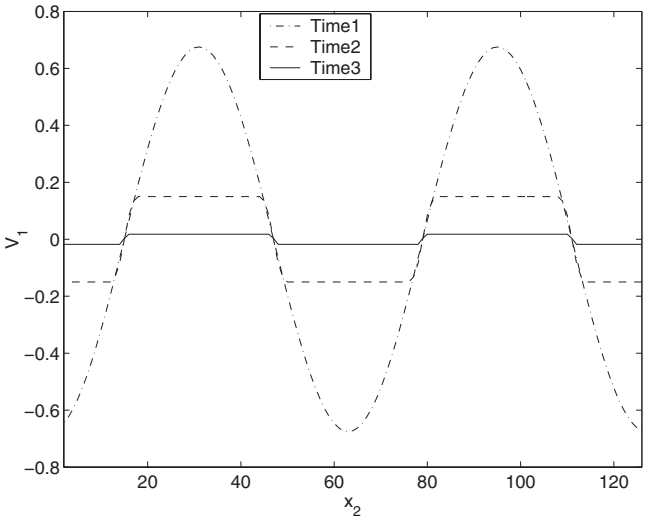

FIG. 2. Evolution of $v_1(x_2,t)$. Note the quick full depression of that component when the scale l_1 along x_1 is much smaller than the scale l_2 along x_2.

tion, we will consider the intermediate stage of evolution of an anisotropic random field under the two-dimensional Burgers equation. Let us assume that the initial potential $\psi_0(x_1,x_2)$ is random and strongly anisotropic field with the spatial scales $l_1 \ll l_2$. The initial energy $E_i(t)=\langle v_i^2\rangle = \langle(\partial\psi/\partial x_i)^2\rangle = \sigma_\psi/l_i^2$ for the velocity component v_1, $E_1(0)$, is in this case much larger than the energy of the large scale component v_2, $E_2(0)$. We can introduce the nonlinear time of i-th component as $t_{\mathrm{nl},i}=l_i^2/\sigma_\psi$. For $t \ll t_{\mathrm{nl},2}$, the drift of the Lagrangian particles in direction x_2 is relatively small. Then one can assume $y_2=x_2$ in Eq. (9) and consider the one-dimensional problem with the initial potential $\psi_0(y_1,x_2,t)$, where x_2 is a parameter.

Due to the condition $l_1 \ll l_2$, the first shock lines in the Lagrangian coordinates are on the points where $\partial\psi/\partial y_1$ has a minimum. In Eulerian space, the shocks are oriented primarily along the x_2 axis and their length in this direction increase in time. For $t \gg t_{\mathrm{nl},1}$, the velocity field $v_1(x_1,x_2,t)$ transforms to the sequence of triangular pulses

$$v_1(x_1,x_2,t) = \frac{x_1 - y_{1,k}(x_1,x_2,t)}{t},$$

$$\text{for} \quad x_{1,k}^s < x_1 < x_{1,k+1}^s, \tag{39}$$

where $y_{1,k}(x_1,x_2,t)$ is the coordinate of the absolute maximum of Eq. (9) over y_1 with $y_2=x_2$. The shock positions $x_{1,k}^s(x_2,t)$ are

$$x_{1,k}^s = \frac{y_{1,k+1}+y_{1,k}}{2} + v_k t,$$

$$v_k = \frac{\psi_0(y_{1,k}(x_1,x_2),x_2) - \psi_0(y_{1,k+1}(x_1,x_2),x_2)}{y_{1,k+1}(x_1,x_2) - y_{1,k}(x_1,x_2)}. \tag{40}$$

It means that at fixed x_2 the interval $x_{1,k}^s < x_1 < x_{1,k+1}^s$ will be covered by the particles from a small region near the Lagrangian point $y_{1,k}(x_1,x_2,t)$, and for the velocity component v_2, we get

$$v_2(x_1,x_2,t) = -\left.\frac{\partial\psi_0(x_1,x_2)}{\partial x_2}\right|_{x_1=y_{1,k}(x_2,t)},$$

$$\text{for} \quad x_{1,k}^s < x < x_{1,k+1}^s. \tag{41}$$

The velocity $v_2(x_1,x_2,t)$ does not depend on x_1 between the shock-lines $x_{1,k}^s(x_2)$ and $x_{1,k+1}^s(x_2)$. The collision of the shocks in a one-dimensional Burgers equation is equivalent to the fact that at some point $\mathbf{x}_*$ two adjacent shock lines $x_{1,k}^s$ and $x_{1,k+1}^s$ touch each other. Then this point will be developed into new shock lines $x_{1,k}^{s,*}(x_2)$ with a length along axis x_2 increasing in time and whose ends transforms into lines $x_{1,k}^s$,

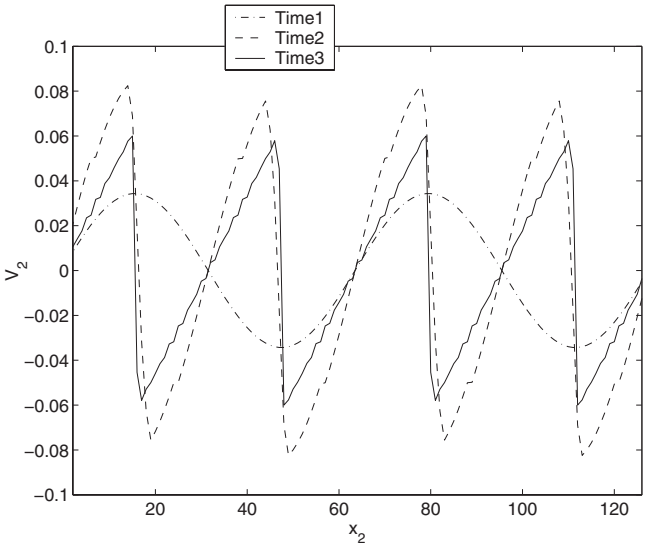

FIG. 3. Evolution of $v_2(x_2,t)$. This component doubles its frequency along x_2 and actually *gains* energy for moderate times.

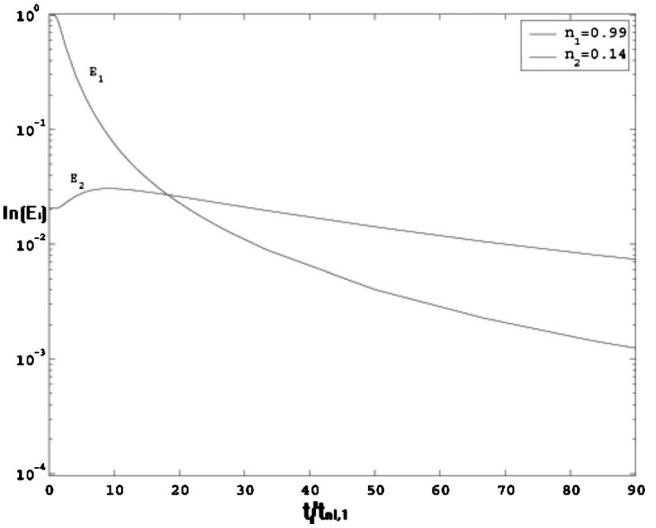

FIG. 4. Evolution of the two velocity components relative energies. The energy E_2 of component 2 actually *increases* for some time and decays much slower than the energy E_1 of the first component, when the two energies are initially quite different, showing that the two components can exchange energy between themselves.

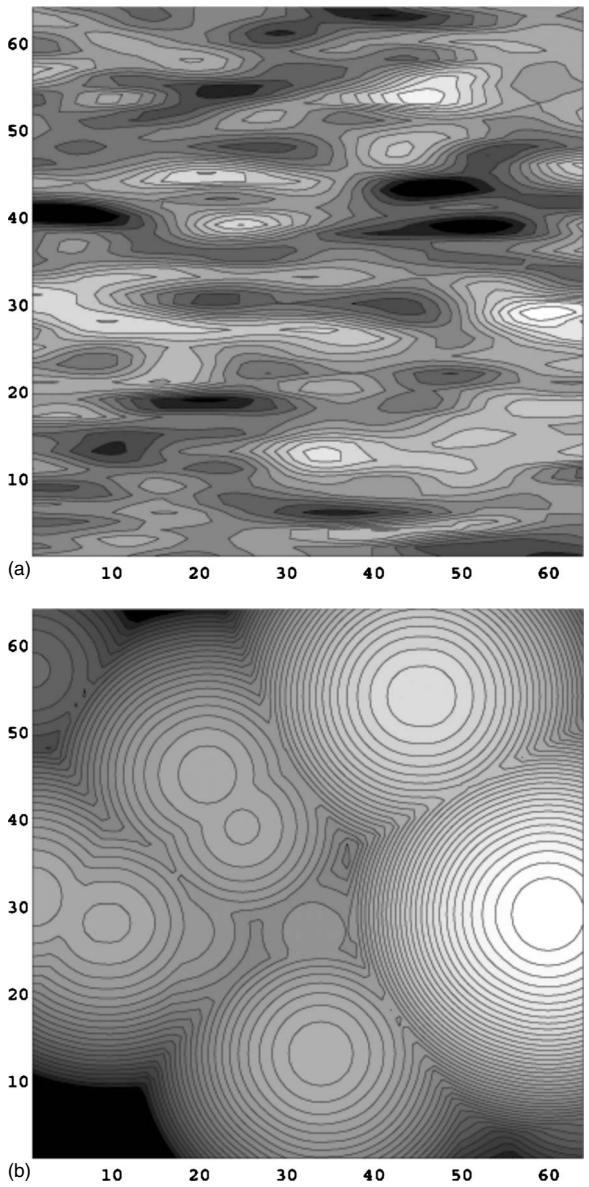

FIG. 5. Evolution of potential for an initially anisotropic random field. The first figure is the initial potential $\psi_0(x_1,x_2)$ and the second is the potential $\psi(x_1,x_2,t)$ at late times when the stage of isotropization has been reached.

$x^s_{1,k+1}$. Thus at $t_{n2,1} \gg t \gg t_{\mathrm{nl},1}$ the velocity field will have a cellular structure, the borders $x^s_{1,k}(x_2)$ of the cells obeying Eq. (40), so that the velocity component $v_1(\mathbf{x})$ will have a universal structure (39). Velocity component $v_2(\mathbf{x})$ inside the cell does not depend on x_1 and along the axis x_2 reproduces the behavior of v_2 along the line $x_1 = y_{1,k}(x_2,t)$ (41). The evolution of the potential is plotted on Fig. 5.

The statistical problems of the velocity component v_1 at this stage are similar to the properties of one-dimensional Burgers turbulence. The integral scale $L_1(t)$ and the energy $E_1(t) \sim \sigma_\psi / t$ of the component v_1 are described by the expressions (2), where σ_ψ is the variance of its initial two-dimensional potential ψ_0 and $l_0 = l_1$ is the integral scale of v_1 component $L_1 = \sigma_\psi / \sigma_{v_1}$. Due to the merging of the shock lines the integral scale of the turbulence along the axis x_1 increases with time $L_1(t) \sim (t\sigma_\psi)^{1/2}$ and at $t \sim t_{\mathrm{nl},2}$, when $L_1(t) \simeq l_2$ and $E_1(t) \simeq E_2(0)$ we need to take into account the nonlinear distortion along the axis x_2. At $t \gg t_{\mathrm{nl},2}$ the potential and velocity fields have a universal isotropic and self-similar structure inside the cells [Eqs. (13) and (14)]. The boundary of the cells on this stage degenerate into straight lines (planes, in three dimensional case). The multiple merging of the cells will lead to the establishment of statistical self-similarity and isotropization of the field. In the next section, we will show how the statistical properties of the isotropic multidimensional Burgers turbulence are connected with the parameter of the anisotropic initial perturbation.

B. Isotropisation of the multidimensional Burgers turbulence

The statistical properties of the Burgers velocity field $\mathbf{v}$ [Eq. (10)] in the limit $\nu \to 0$ are determined by the statistical properties of the absolute maxima coordinate $\mathbf{y}(\mathbf{x},\mathbf{t})$ of the function (9). In the one-dimensional case, the problem of the absolute maximum is reduced to the problem of finding the crossing of the random signal $\psi_0(x)$ by the nonhomogeneous function $(x-y)^2/2t+H$. The asymptotic behavior of the field at large t is determined by the maximum whose amplitude is higher than the variance of the initial potential. That is why one can use some results of the theory of extremal processes [3,35,40]. In the multidimensional case, the problem of the peaks statistics of the Gaussian field is rather well-known for the isotropic and homogeneous field [41]. But for the Burgers turbulence, we need to find the statistical properties of the absolute maximum of the nonhomogeneous and anisotropic scalar field $\Phi(\mathbf{x},\mathbf{y},t)$. In paper [42], it was shown that at large time t this problem is reduced to the problem of finding the statistical properties of extrema of random field $\psi_0(\mathbf{x})$ whose values are much greater than the variance σ_ψ.

Let us assume the initial potential $\psi_0(\mathbf{x})$ is a random Gaussian field and its correlation function has the following form:

$$\langle \psi_0(\mathbf{x})\psi_0(\mathbf{x}+\mathbf{z})\rangle = B_\psi(\mathbf{z}) = \sigma_\psi^2 \prod_{i=1}^{d} R_i(z_i), \tag{42}$$

$$R_i(z_i) = 1 - \frac{z_i^2}{2!l_{0,i}^2} + \frac{z_i^4}{4!l_{1,i}^4} + \ldots \tag{43}$$

We assume also that the correlation function decreases rather fast at large distances $B_\psi(\mathbf{z}) \ll \sigma_\psi^2$ when $|\mathbf{z}| > l_{st}$. Then the values of the Gaussian initial field $\psi_0(\mathbf{x})$ will be statistically independent for two points $\mathbf{x}_1$ and $\mathbf{x}_2$ such that their distance is larger than l_{st}, $|\mathbf{x}_1 - \mathbf{x}_2| > l_{st}$.

In the limit of vanishing viscosity, we have "maximum representation" (9) for the solution of the potential. In this solution the velocity field $\mathbf{v}(\mathbf{x},t)$ (10) is determined by the coordinate $\mathbf{y}$ of the absolute maximum of function $\Phi(\mathbf{x},\mathbf{y},t)$. Let $Q(H,\Delta V_k)$ denote the cumulative probability and $W_{\max}(H,\Delta V_k)$ denote the probability density of the absolute maximum in an elementary volume ΔV_k

$$Q(H,\Delta V_k) = \mathrm{Prob}(\Phi < H, \mathbf{y} \in \Delta V_k), \tag{44}$$

$$W_{\max}(H,\Delta V_k) = Q'_H(H,\Delta V_k). \tag{45}$$

Here, we introduce the elementary volume ΔV_k whose scale is much greater than l_{st}, but much smaller than the integral scale of turbulence $L(t)$. The probability for the absolute maximum to be contained between H_1 and $H_1+\Delta H_1$ with the coordinate $\mathbf{y}(\mathbf{x},t) \in \Delta V_k$ equals to that one for the absolute maximum $H_1 \in (H, H+\Delta H)$ to lie in ΔV_k and to be less than H in magnitude for outer intervals $\overline{\Delta V_k}$

$$\begin{aligned}\mathrm{Prob}(\mathbf{y} \in \Delta V_k, H \in [H_1, H_1+\Delta H]) \\ = W_{\max}(H,\Delta v_k)\Delta H Q(H,\overline{\Delta V_k}).\end{aligned} \tag{46}$$

Here, we propose that the absolute maximums are statistically independent in the intervals ΔV_k and $\overline{\Delta V_k}$. The probability for coordinate $\mathbf{y}(\mathbf{x},t)$ to fall into ΔV_k can be obtained by the integration of Eq. (46) with respect to H

$$\mathrm{Prob}(\mathbf{y} \in \Delta V_k) = \int W_{\max}(H,\Delta V_k) Q(H,\overline{\Delta V_k}) dH. \tag{47}$$

After the integration of Eq. (47) by parts, we have

$$\mathrm{Prob}(\mathbf{y} \in \Delta V_k) = \int N(H,\Delta V_k) Q'_H(H) dH, \tag{48}$$

where $Q(H)$ is the integral distribution function of the absolute maximum in the whole space. In Appendix in [42], it was shown that for large H the integral distribution function

$$Q(H,\Delta V_k) = \exp[-N(H,\Delta V_k)] \tag{49}$$

is determined by the mean number of extrema $N(H,\Delta V_k)$ with value larger than H.

Let us first consider the statistical properties of the extremum of the homogeneous random field $\psi_0(\mathbf{x})$. It is known that for smooth fields, the number of crossings of some high level asymptotically tends to the number of maxima and number of extrema. It means that all the peaks above some high level have only one extremum, which is the maximum of this peak. Thus, we will consider first the properties of extremum of the field $\psi_0(\mathbf{x})$. Using these properties of δ function one can obtain for the mean number $N(H;V) = \langle N_{\mathrm{exp}} \rangle$ of extremum with the value greater than H in some volume V

$$N(H;V) = \left\langle \int_V \delta(\nabla \psi_0) |J(a_{ij})| E(\psi_0 - H) d\mathbf{x} \right\rangle. \tag{50}$$

Here, $E(s)$ is the Heaviside unit function, and J is the Jacobian of the transformation

$$J(a_{ij}) = |a_{ij}|, \quad a_{ij} = \frac{\partial \psi_0(\mathbf{x})}{\partial x_i \partial x_j}. \tag{51}$$

For the homogeneous field $\psi_0(\mathbf{y})$, one can introduce the density of extremum as $n_{\mathrm{ext}}(H;V) = N(H)/V$. The density of n_{ext} in this case is determined by the joint probability distribution function of the ψ_0, their gradient $v_i = \partial \psi_0 / \partial x_i$ and tensor $a_{ij} = \partial \psi_0 / \partial x_i \partial x_j$. For the homogeneous Gaussian field $W_{\psi_0, v_i, a_{ij}} = W_v(v_i) W_{\psi_0, a_{ij}}(\psi_0, a_{ij})$ and from Eq. (50), we have for the density of extremum

$$n_{\mathrm{ext}}(H) = W_v(0) \int_H^\infty dS \int da_{ij} J(a_{ij}) W_{\psi_0, a_{ij}(S, a_{ij})}. \tag{52}$$

For the Gaussian field, the probability density function (p.d.f.) of the field $\psi_0(\mathbf{x})$ and its derivative are determined by the correlation function of $\psi_0(\mathbf{x})$ (42). In Eq. (52), we will integrate over the conditional probability $W_{\mathrm{con}}(a_{ij}/S) = W(a_{ij}, S)/W_{\psi_0}(S)$ and will get

$$n_{\mathrm{ext}} = W_v(0) \int_H^\infty dS W_{\psi_0}(S) \int da_{ij} J(a_{ij}) W_{\mathrm{con}}(a_{ij}/S). \tag{53}$$

Using the properties of Gaussian variables, one can receive that the conditional expected value of a_{ij} is $\langle a_{ij} \rangle_{\mathrm{con}} = S\langle a_{ij} S \rangle / \sigma_\psi^2$. In the problem of the Burgers turbulence at large time the asymptotic of n_{ext} of high value H is important. Thus in conditional averaging we have

$$\langle J(a_{ij}) \rangle_{\mathrm{con}} \simeq J(\langle a_{ij} \rangle_{\mathrm{con}}) \simeq \prod_{i=1}^{d} \langle a_{ii} \rangle_{\mathrm{con}} = \frac{S^d}{l_{0,\mathrm{eff}}^{2d}}. \tag{54}$$

Here, we introduce the effective length $l_{0,\mathrm{eff}}$

$$l_{0,\mathrm{eff}}^d = \prod_{i=1}^{d} l_{0,i} \tag{55}$$

and take into account that $\langle a_{ii}^2 \rangle = \sigma_\psi^2 / l_{0,i}^4$. Finally, we obtain from Eqs. (53) and (54) for the density of extremum the following expression:

$$\begin{aligned} n_{\mathrm{ext}}(H) &= W_v(0) \int_H^\infty dS W_{\psi_0}(S) \frac{S^d}{l_{\mathrm{eff}}^{2d}} \\ &= \frac{1}{(2\pi)^{d+1/2} l_{\mathrm{eff}}^d} \int_{H/\sigma_\psi}^\infty S^d e^{-S^2/2\sigma_\psi^2} dS \\ &\simeq \left(\frac{H}{\sigma_\psi}\right)^{d-1} \frac{1}{(2\pi)^{d+1/2} l_{\mathrm{eff}}^d} e^{-H^2/2\sigma_\psi^2}. \end{aligned} \tag{56}$$

Thus, from Eq. (56), one gets that the mean number of extrema of an anisotropic field $\psi_0(\mathbf{x})$ is determined by some effective spatial scale l_{eff}, which is the geometrical mean of the spatial scales $l_{0,i}$. For relatively large values of H, the density of extremum (56) is equal to the density of events where the random field $\psi_0(\mathbf{x})$ is larger than H.

For the homogeneous field $N(H;V) = V n_{\mathrm{ext}}(H)$, where $n_{\mathrm{ext}}(H)$ is described by the Eq. (56). For the nonhomogeneous field, even in one-dimensional case, the expression for N is more complicated. We assume that the nonhomogeneous field is $\Phi(\mathbf{x}) = \psi_0(\mathbf{x}) - \alpha(\mathbf{x})$ and $\alpha(\mathbf{x})$ is a smooth function in scale of $\psi_0(\mathbf{x})$. Then in a quasistatic approximation one can receive for the mean number of events $N(H;V)$ such that $\Phi(\mathbf{x}) > H$ in a volume V the following expression:

$$N(H;V) = \int_V n_{\mathrm{ext}}[H + \alpha(\mathbf{x})] dV, \tag{57}$$

where $n_{\mathrm{ext}}(H)$ is determined by the expression (56) and is the density of extremum of the statistically homogeneous function $\psi_0(\mathbf{x})$.

At large time, the paraboloid $\alpha=(\mathbf{x}-\mathbf{y})^2/2t$ in Eq. (9) is a smooth function in the scale of the initial potential. Then for the mean number of maxima, one can use a quasi-static approximation (57) and

$$Q(H)=\exp[-N_\infty(H)] \tag{58}$$

$$N_\infty(H)=\int n_{\mathrm{ext}}\left(H+\frac{(\mathbf{x}-\mathbf{y})^2}{2t}\right)dV. \tag{59}$$

Here, $N_\infty(H)$ is the mean number of extrema of $\Phi(\mathbf{x},\mathbf{y},t)$ in the whole space with magnitude greater than H and $n_{\mathrm{ext}}(H)$ is the density of the number of extremum of the initial homogeneous potential $\psi_0(\mathbf{x})$ with value greater than H. For $H \gg \sigma_\psi$ the density $n_{\mathrm{ext}}(H)$ is determined by the expression (56) and we have

$$N_\infty(H)=\frac{1}{(2\pi)^{(d+1)/2}l_{\mathrm{eff}}^d}\int\left(\frac{H}{\sigma_\psi}\right)^{d-1}e^{-(H+\mathbf{y}^2/2t)^2/2\sigma_\psi^2}d\mathbf{y}$$
$$\simeq\left(\frac{H}{\sigma_\psi}\right)^{d-1}\frac{1}{\sqrt{2\pi}}\left(\frac{\sigma_\psi^2 t}{Hl_{\mathrm{eff}}^2}\right)^{d/2}e^{-H/2\sigma_\psi^2}. \tag{60}$$

In Eq. (58), we integrate over the infinite space, but the effective volume $(\sigma_\psi^2 t/Hl_{\mathrm{eff}}^2)^{d/2}$ is determined by the paraboloid term in Eq. (58). Now the effective number of independent local maximum in initial perturbation is $N_{\max}\sim(\sigma_\psi t/l_{\mathrm{eff}}^2)^{d/2}$ and increases with time. When $N_{\max}\gg 1$, we can introduce the dimensionless potential h as follows:

$$H=h\sigma_\psi,\quad h=h_0(1+z/h_0^2), \tag{61}$$

where $h_0=H_0\sigma_\psi$ and H_0 is the solution of the equation $N(H_0)=1$

$$h_0\simeq d^{1/2}\left(\log\frac{\sigma_\psi t}{l_{\mathrm{eff}}^2(2\pi)^{1/d}}\right)^{1/2},\quad \langle\psi(\mathbf{x},t)\rangle\simeq\sigma_\psi h_0. \tag{62}$$

Thus, we have a logarithmic growth of mean potential. The dimensionless potential has a double exponential distribution

$$Q(z)=\exp(-\exp(-z)),$$

$$Q_h(h)=\exp(-\exp(-(h-h_0)h_0), \tag{63}$$

One can see that for $t\gg t_{\mathrm{nl}}=l_{\mathrm{eff}}^2/\sigma_\psi$, we have $N_{\max}\gg 1$ and the integral distribution of absolute maximum is concentrated in the narrow region $\Delta H/H\simeq\sigma_\psi^2/H_0^2\ll 1$ near H_0. Using this fact, one can get from Eq. (48) the probability distribution of the maximums coordinate

$$W(\mathbf{y},\mathbf{x},t)=\frac{1}{\sqrt{2\pi L^2(t)}}\exp-\frac{(\mathbf{x}-\mathbf{y})^2}{2L^2(t)}, \tag{64}$$

where

$$L(t)=\left(\frac{\sigma_\psi t}{h_0}\right)^{1/2}=(\sigma_\psi t)^{1/2}d^{-1/4}\left(\log\frac{\sigma_\psi t}{l_{\mathrm{eff}}^2(2\pi)^{1/d}}\right)^{-1/4} \tag{65}$$

is the integral scale of the turbulence. From Eqs. (10) and (64), we see that the one-dimensional probability distribution of the velocity field is Gaussian and isotropic. For the energy of each component, we have

$$E_i(t)=\sigma_{v,i}^2=\frac{L^2(t)}{t^2}=\left(\frac{\sigma_\psi}{t}\right)d^{-1/2}\left(\log\frac{\sigma_\psi t}{l_{\mathrm{eff}}^2(2\pi)^{1/d}}\right)^{-1/2}. \tag{66}$$

Thus, for the anisotropic initial field there is the isotropisation of the turbulence.

For the multidimensional Burgers turbulence, the two-dimensional probability distribution, correlation function and energy spectrum where found in [3,16] using so-called "cellular" model. In this model, it is assumed that in different elementary volumes, the initial potential are independent and that the potential has a Gaussian distribution. In this model, there is a free parameter Δ, which is the size of the elementary cell. In the present work, we consider a continuous initial random potential field with given correlation function (42) and (43). The procedure for calculation of the two-point probability distribution function is nevertheless similar to that one used in [3,16]. It is easy to show that for the two-point p.d.f., we have the same expression as obtained in [3] for the cell model, but with a size of the cell Δ changing with the effective spatial scale l_{eff}. The effective spatial scale $l_{0,\mathrm{eff}}$ (55) is determined by the scales $l_{0,i}$ of its initial correlation function by Eq. (42). For the two-point p.d.f., correlation function, and energy spectrum, we also have self-similarity and isotropisation at large times. In particular for the normalized longitudinal and transverse correlation function of the velocity field $\tilde{\mathbf{v}}=\mathbf{v}/\sigma_{v,i}$, we have

$$\tilde{B}_{LL}(\tilde{x})=\langle\tilde{v}_{1L}\tilde{v}_{2L}\rangle=\frac{d}{d\tilde{x}}[\tilde{x}P(\tilde{x})], \tag{67}$$

$$\tilde{B}_{NN}(\tilde{x})=\frac{1}{2}\langle\tilde{v}_{1N}\tilde{v}_{2N}\rangle=P(\tilde{x}), \tag{68}$$

where $\tilde{x}=x/L(t)$ and

$$P(\tilde{x})=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dz}{g\left(\frac{\tilde{x}+z}{2}\right)\exp\left[\frac{(\tilde{x}+z)^2}{8}\right]+g\left(\frac{\tilde{x}-z}{2}\right)\exp\left[\frac{(\tilde{x}-z)^2}{8}\right]}, \tag{69}$$

$$g(z) \equiv \int_{-\infty}^{z} e^{-s^2/2} ds. \tag{70}$$

It may be shown that the function $P(\tilde{x})$ is the probability of having no shock within an Eulerian interval of length $\tilde{x}L(t)$. As far as potential isotropic fields are concerned, the normalized energy spectrum $e(k)$ is formulated via a one-dimensional spectrum $e_{NN}(k)$ of the transverse component. The energy spectrum $E_v(k,t)$ is then isotropic and self-similar

$$E(k,t) = \frac{L^3(t)}{t^2} \tilde{E}[kL(t)]. \tag{71}$$

At large wave number k, the discontinuity initiation leads to the power asymptotic behavior $E(k,t) \sim k^{-2}$. At small wave number, one also has a universal behavior

$$E(k,t) = k^{d+1} \frac{L^{4+d}(t)}{t^2} \sim k^{d+1} t^{d/2}, \tag{72}$$

which has to do with the nonlinear generation of low-frequency component. In particular for the three-dimensional turbulence we have $E(k,t) \sim k^4 t^{3/2}$. For large, but finite Reynolds numbers, the "shocks" have a finite width $\delta \sim \nu t / L(t)$ and their relative width increases slowly with time $\delta / L(t) \sim (\log(\sigma_\psi t / l_{\text{eff}}^2))^{1/2}$. Thus, at very large time, we have a linear stage of evolution.

C. Linear stage of evolution of Burgers turbulence

Let us now consider the linear stage of evolution of a random field, when the potential $\psi(\mathbf{x},t)$ and the velocity field $\mathbf{v}(\mathbf{x},t)$ (16) are linearly connected with the solution $u(\mathbf{x},t)$ of the linear diffusion Eq. (7). Here, $u(\mathbf{x},t)$ is the relative fluctuation of the field $U(\mathbf{x},t)$ (15). Introduce the spectral density of the field $u(\mathbf{x},t)$ as

$$E_u(\mathbf{k},t) = \frac{1}{(2\pi)^d} \int B_u(\mathbf{z},t) e^{i(\mathbf{kz})} d\mathbf{z}, \tag{73}$$

where $B_u(\mathbf{z},t) = \langle u(\mathbf{z},t) u(0,t) \rangle$ is a correlation function of relative fluctuation field $\tilde{U}(\mathbf{x},t)$. The evolution of the spectral density $E_u(\mathbf{k},t)$ and variance $\sigma_u^2(t)$ of u are described by the equations

$$E_u(\mathbf{k},t) = E_{u0}(\mathbf{k}) e^{-2\nu k^2 t}, \quad E_{u0}(\mathbf{k}) = E_u(\mathbf{k},0), \tag{74}$$

$$\sigma_u^2(t) = \int E_{u0}(\mathbf{k}) e^{-2\nu k^2 t} d\mathbf{k}. \tag{75}$$

For homogeneous Gaussian initial potentials $\psi_0(\mathbf{x})$, we have from (6)

$$E_{u0}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int \left[\exp\left(\frac{B_\psi(\mathbf{z})}{4\nu^2} \right) - 1 \right] e^{i(\mathbf{kz})} d\mathbf{z}, \tag{76}$$

where $B_\psi(\mathbf{z})$ is the correlation function of the initial potential $\psi_0(\mathbf{x})$. The condition for Burgers turbulence to enter the linear regime is $t \gg t_{\text{lin}}$, where t_{lin} is determined from the equation $\sigma_u^2(t_{\text{lin}}) \simeq 1$. From Eq. (74), one can see that the large times behavior of the scalar field u and consequently the velocity field $\mathbf{v}$ will be determined by the behavior of the energy spectrum $E_{u0}(\mathbf{k})$ at small wave numbers $\mathbf{k}$. When the correlation function of initial potential $\psi_0(\mathbf{x})$ may be represented in the form (42) at small ρ and $B_\psi(\rho|) \ll 1$ for $|\rho| > l_{st}$, then from Eq. (76), we have that the spectrum $E_{u0}(\mathbf{k})$ at $k \ll \text{Re}_0 / l_{\text{eff}}$ is flat and

$$E_{u0}(\mathbf{k}=0) = D_u \simeq \frac{1}{(2\pi)^{d/2}} (l_{\text{eff}} / \text{Re}_0)^d \exp(\text{Re}_0^2), \tag{77}$$

where l_{eff} is determined by Eq. (55). The spectrum of the field u at large time is isotropic and has an universal form

$$E_u(\mathbf{k},t) = D_u e^{-2\nu k^2 t} \tag{78}$$

and consequently isotropic is the velocity field [3,16,27]. The energy of each component decays as $E_i(t) \sim D_u (\nu t)^{-(d+2)/2}$.

Consider now the case when the initial potential has a correlation function of the form (42) at small $\mathbf{x}$ but has long range correlations extending to large $\mathbf{x}$ [27]

$$B_\psi(\mathbf{x}) = \sigma_\psi^2 (|\mathbf{x}| / l_{\text{long}})^{-\alpha} F_B(\mathbf{x}/|\mathbf{x}|), \quad \alpha > 0. \tag{79}$$

Here, the function $F_B(\mathbf{x}/|\mathbf{x}|)$ describes the anisotropy of correlation function at long distances. If $\alpha > d$ the evolution of the spectrum of u Eq. (78) and of the velocity $\mathbf{v}$ will be the same as in the absence of long range correlations [27]. At $0 < \alpha < d$, the energy spectrum of initial potential has singularities at small wave number

$$E_\psi(\mathbf{k}) = \sigma_\psi^2 l_{\text{long}}^\alpha |\mathbf{k}|^{\alpha-d} F_E(\mathbf{k}/|\mathbf{k}|), \quad 0 < \alpha < d, \tag{80}$$

where the function $F_E(\mathbf{k}/|\mathbf{k}|)$ is determined by the function $F_B(\mathbf{x}/|\mathbf{x}|)$ and describes the anisotropy of potential spectrum at small wave number. It was shown [27] that in this case we have the conservation of anisotropy at linear stage and the asymptotic behavior of the spectrum of u is determined by the Eq. (74) where

$$E_u(\mathbf{k}) = E_\psi(\mathbf{k}) / (4\nu)^2 = \text{Re}_0^2 \, l_{\text{long}}^\alpha |\mathbf{k}|^{\alpha-d} F_E(\mathbf{k}/k). \tag{81}$$

For the velocity spectrum, it means that it reproduces the initial spectrum of velocity at small wave number multiplied by the exponential factor $\exp(-2\nu k^2 t)$. These results were formulated in [27] for the correlation function of velocity fields. In the paper [27] it was also shown that asymptotically the velocity field has a Gaussian distribution. But the transformation processes to the linear stage are not trivial and have to be estimated in the spectral representation. The behavior of the spectral density of the field u at small wave number $\mathbf{k}$ is determined by the tail of correlation function $B_\psi(\mathbf{x})$ Eq. (79) and from Eq. (76), we have that the spectrum is described by the Eq. (81). But at higher values of the modulus of the wave number k, the anisotropic power spectrum is transformed to the flat spectrum (77). The wave number k_{af} where this transformation takes place may be estimated from the condition that at $k \simeq k_{af}$ these spectrum are of the same order and we have

$$k_{af} \simeq \text{Re}_0^{(d+2)/(d-\alpha)} \left(\frac{l_{\text{long}}^\alpha}{l_{\text{eff}}^d} \right)^{1/(d-\alpha)} e^{-\text{Re}_0^2/(d-\alpha)}. \tag{82}$$

The condition for Burgers turbulence to enter the linear regime is now determined from the equation $\sigma_u^2(t_{\rm lin}) \simeq 1$. The main contribution in the variance $\sigma_u^2(t)$ (75) is from the flat spectrum (77). The wave number $k_{\rm lin} \simeq (\nu t_{\rm lin})^{-1/2}$ is

$$k_{\rm lin} \simeq ({\rm Re}_0/l_{\rm eff}) e^{-{\rm Re}_0^2/d}. \quad (83)$$

Thus, for the ratio of these two critical wave numbers we have

$$k_{af}/k_{\rm lin} \simeq {\rm Re}_0^{(\alpha+2)/(d-\alpha)} \left(\frac{l_{\rm long}}{l_{\rm eff}}\right)^{1/(d-\alpha)} e^{-{\rm Re}_0^{2\,\alpha/d(d-\alpha)}}, \quad (84)$$

and for the large initial Reynolds number $k_{af} \ll k_{\rm lin}$. From the equation $\sigma_u^2(t_{\rm lin}) \simeq 1$, we have that $t_{\rm lin} \simeq t_{\rm nleff}\, {\rm Re}_0^{-1} \exp(2\, {\rm Re}_0^2)$ and is extremely large in comparison with the effective nonlinear time $t_{\rm nleff} \simeq l_{\rm eff}^2/\sigma_\psi$. It is easy to see from Eq. (82) that the time of "isotropization" $t_{\rm iso} \simeq \nu k_{af}^2$ is much greater than the nonlinear time $t_{\rm iso} \sim t_{\rm nleff} e^{2\,{\rm Re}_0^{2\,\alpha/d(d-\alpha)}}$ and this difference increases when the index α approaches the critical value $\alpha \simeq d$.

V. DISCUSSION AND CONCLUSION

Let us now discuss the evolution of the multidimensional Burgers turbulence in presence of anisotropy at small Eq. (42) or large Eq. (79) spatial scales. In the initial perturbation, the energy of the velocity component in direction i is $\sigma_\psi^2/l_{0,i}^2$ and is greater for directions having smaller initial characteristic scales $l_{0,i}$. At the initial stage, the scale of the turbulence in this direction will increase faster than in the others and we have the energy decay primarily of this component (see Sec. IV A). When the time t becomes greater than the nonlinear time of the component with the largest scale $t_{{\rm nl},i} = \max l_{0,i}^2/\sigma_\psi$, we will have the isotropization of the turbulence for scales of the order of the integral scale $L(t)$ (65). In Sec. IV B, we did consider the situation in the absence of long scale correlation. But based on the results of the one-dimensional case [36], we may suggest that there is no influence of a long scale correlation on the evolution of the energy. Nevertheless, we still have conservation of the anisotropy at large scales $|\mathbf{x}| \gg L(t)$ (79). In the spectral representation, we have conservation of the initial spectrum and anisotropy at small wave number, but at $k \sim k_s(t) \sim t^{-p}$, the initial spectrum transforms into a self-similar spectrum (71) with the universal behavior $E(k,t) \sim k^{(d+1)}$ (72) for $kL(t) < 1$. Let us define an energy wave number $k_L(t) = L^{-1}(t) \sim (t\sigma_\psi)^{-1/2}$, which is roughly the wave number around which most of the kinetic energy resides. Hence, the switching wave number $k_s(t)$ goes to zero much faster than the energy wave number. If we take into account the finite viscosity, we have that at times $t \gg t_{\rm lin} \simeq t_{\rm nleff}\, {\rm Re}_0^{-1} \exp(2\, {\rm Re}_0^2)$, the nonlinear evolution of the spectrum becomes frozen and only the linear decay of the small scales is significant Eq. (74). The frozen spectrum of the velocity potential has a critical wave number k_{af} below which the spectrum is anisotropic and reproduces the initial spectrum, and for $k > k_{af}$ the spectrum is flat Eq. (77). Thus, at times $t_{af} \gg t \gg t_{\rm lin}$, this part of the spectrum will play the dominant role in the evolution of the velocity spectrum, and consequently the energy of all velocity components will be equal. At times $t \gg t_{af}$, the spectrum of the velocity will reproduce the small scale part of the initial velocity spectrum, but multiplied by an exponential factor Eq. (74), and will in the end be an anisotropic field.

ACKNOWLEDGMENTS

We have benefited from discussion with U. Frisch, A. Saichev, and A. Sobolevskii. This paper also improved very much thanks to comments and corrections of J.-D. Fournier. This work was supported by Grants No. RFBR-08-02-00631-a and No. SS 3700.2010.2. S. Gurbatov thanks the French Ministry of Education, Research and Technology for support during his visit to the Observatoire de la Côte d'Azur.

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